

# SEMI-FREDHOLM SINGULAR INTEGRAL OPERATORS WITH PIECEWISE CONTINUOUS COEFFICIENTS ON WEIGHTED VARIABLE LEBESGUE SPACES ARE FREDHOLM

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**ABSTRACT.** Suppose  $\Gamma$  is a Carleson Jordan curve with logarithmic whirl points,  $\varrho$  is a Khvedelidze weight,  $p : \Gamma \rightarrow (1, \infty)$  is a continuous function satisfying  $|p(\tau) - p(t)| \leq -\text{const}/\log|\tau - t|$  for  $|\tau - t| \leq 1/2$ , and  $L^{p(\cdot)}(\Gamma, \varrho)$  is a weighted generalized Lebesgue space with variable exponent. We prove that all semi-Fredholm operators in the algebra of singular integral operators with  $N \times N$  matrix piecewise continuous coefficients are Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ .

## 1. INTRODUCTION

Let  $X$  be a Banach space and  $\mathcal{B}(X)$  be the Banach algebra of all bounded linear operators on  $X$ . An operator  $A \in \mathcal{B}(X)$  is said to be  $n$ -normal (resp.  $d$ -normal) if its image  $\text{Im } A$  is closed in  $X$  and the defect number  $n(A; X) := \dim \text{Ker } A$  (resp.  $d(A; X) := \dim \text{Ker } A^*$ ) is finite. An operator  $A$  is said to be semi-Fredholm on  $X$  if it is  $n$ -normal or  $d$ -normal. Finally,  $A$  is said to be Fredholm if it is simultaneously  $n$ -normal and  $d$ -normal. Let  $N$  be a positive integer. We denote by  $X_N$  the direct sum of  $N$  copies of  $X$  with the norm

$$\|f\| = \|(f_1, \dots, f_N)\| := (\|f_1\|^2 + \dots + \|f_N\|^2)^{1/2}.$$

Let  $\Gamma$  be a Jordan curve, that is, a curve that is homeomorphic to a circle. We suppose that  $\Gamma$  is rectifiable. We equip  $\Gamma$  with Lebesgue length measure  $|d\tau|$  and the counter-clockwise orientation. The *Cauchy singular integral* of  $f \in L^1(\Gamma)$  is defined by

$$(Sf)(t) := \lim_{R \rightarrow 0} \frac{1}{\pi i} \int_{\Gamma \setminus \Gamma(t, R)} \frac{f(\tau)}{\tau - t} d\tau \quad (t \in \Gamma),$$

where  $\Gamma(t, R) := \{\tau \in \Gamma : |\tau - t| < R\}$  for  $R > 0$ . David [7] (see also [3, Theorem 4.17]) proved that the Cauchy singular integral generates the bounded operator  $S$  on the Lebesgue space  $L^p(\Gamma)$ ,  $1 < p < \infty$ , if and only if  $\Gamma$  is a Carleson (Ahlfors-David regular) curve, that is,

$$\sup_{t \in \Gamma} \sup_{R > 0} \frac{|\Gamma(t, R)|}{R} < \infty,$$

where  $|\Omega|$  denotes the measure of a measurable set  $\Omega \subset \Gamma$ . We can write  $\tau - t = |\tau - t|e^{i\arg(\tau-t)}$  for  $\tau \in \Gamma \setminus \{t\}$ , and the argument can be chosen so that it is

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continuous on  $\Gamma \setminus \{t\}$ . It is known [3, Theorem 1.10] that for an arbitrary Carleson curve the estimate

$$\arg(\tau - t) = O(-\log |\tau - t|) \quad (\tau \rightarrow t)$$

holds for every  $t \in \Gamma$ . One says that a Carleson curve  $\Gamma$  satisfies the *logarithmic whirl condition* at  $t \in \Gamma$  if

$$(1) \quad \arg(\tau - t) = -\delta(t) \log |\tau - t| + O(1) \quad (\tau \rightarrow t)$$

with some  $\delta(t) \in \mathbb{R}$ . Notice that all piecewise smooth curves satisfy this condition at each point and, moreover,  $\delta(t) \equiv 0$ . For more information along these lines, see [2], [3, Chap. 1], [4].

Let  $t_1, \dots, t_m \in \Gamma$  be pairwise distinct points. Consider the Khvedelidze weight

$$\varrho(t) := \prod_{k=1}^m |t - t_k|^{\lambda_k} \quad (\lambda_1, \dots, \lambda_m \in \mathbb{R}).$$

Suppose  $p : \Gamma \rightarrow (1, \infty)$  is a continuous function. Denote by  $L^{p(\cdot)}(\Gamma, \varrho)$  the set of all measurable complex-valued functions  $f$  on  $\Gamma$  such that

$$\int_{\Gamma} |f(\tau)\varrho(\tau)/\lambda|^{p(\tau)} |d\tau| < \infty$$

for some  $\lambda = \lambda(f) > 0$ . This set becomes a Banach space when equipped with the Luxemburg-Nakano norm

$$\|f\|_{p(\cdot), \varrho} := \inf \left\{ \lambda > 0 : \int_{\Gamma} |f(\tau)\varrho(\tau)/\lambda|^{p(\tau)} |d\tau| \leq 1 \right\}.$$

If  $p$  is constant, then  $L^{p(\cdot)}(\Gamma, \varrho)$  is nothing else than the weighted Lebesgue space. Therefore, it is natural to refer to  $L^{p(\cdot)}(\Gamma, \varrho)$  as a *weighted generalized Lebesgue space with variable exponent* or simply as weighted variable Lebesgue spaces. This is a special case of Musielak-Orlicz spaces [24]. Nakano [25] considered these spaces (without weights) as examples of so-called modular spaces, and sometimes the spaces  $L^{p(\cdot)}(\Gamma, \varrho)$  are referred to as weighted Nakano spaces.

If  $S$  is bounded on  $L^{p(\cdot)}(\Gamma, \varrho)$ , then from [13, Theorem 6.1] it follows that  $\Gamma$  is a Carleson curve. The following result is announced in [16, Theorem 7.1] and in [18, Theorem D]. Its full proof is published in [20].

**Theorem 1.1.** *Let  $\Gamma$  be a Carleson Jordan curve and  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying*

$$(2) \quad |p(\tau) - p(t)| \leq -A_{\Gamma} / \log |\tau - t| \quad \text{whenever } |\tau - t| \leq 1/2,$$

*where  $A_{\Gamma}$  is a positive constant depending only on  $\Gamma$ . The Cauchy singular integral operator  $S$  is bounded on  $L^{p(\cdot)}(\Gamma, \varrho)$  if and only if*

$$(3) \quad 0 < 1/p(t_k) + \lambda_k < 1 \quad \text{for all } k \in \{1, \dots, m\}.$$

We define by  $PC(\Gamma)$  as the set of all  $a \in L^{\infty}(\Gamma)$  for which the one-sided limits

$$a(t \pm 0) := \lim_{\tau \rightarrow t \pm 0} a(\tau)$$

exist at each point  $t \in \Gamma$ ; here  $\tau \rightarrow t - 0$  means that  $\tau$  approaches  $t$  following the orientation of  $\Gamma$ , while  $\tau \rightarrow t + 0$  means that  $\tau$  goes to  $t$  in the opposite direction. Functions in  $PC(\Gamma)$  are called piecewise continuous functions.

The operator  $S$  is defined on  $L_N^{p(\cdot)}(\Gamma, \varrho)$  elementwise. We let stand  $PC_{N \times N}(\Gamma)$  for the algebra of all  $N \times N$  matrix functions with entries in  $PC(\Gamma)$ . Writing the elements of  $L_N^{p(\cdot)}(\Gamma, \varrho)$  as columns, we can define the multiplication operator  $aI$  for  $a \in PC_{N \times N}(\Gamma)$  as multiplication by the matrix function  $a$ . Let  $\text{alg}(S, PC; L_N^{p(\cdot)}(\Gamma, \varrho))$  denote the smallest closed subalgebra of  $\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))$  containing the operator  $S$  and the set  $\{aI : a \in PC_{N \times N}(\Gamma)\}$ .

For the case of piecewise Lyapunov curves  $\Gamma$  and constant exponent  $p$ , a Fredholm criterion for an arbitrary operator  $A \in \text{alg}(S, PC; L_N^p(\Gamma, \varrho))$  was obtained by Gohberg and Krupnik [10] (see also [11] and [22]). Spitkovsky [29] established a Fredholm criterion for the operator  $aP + Q$ , where  $a \in PC_{N \times N}(\Gamma)$  and

$$P := (I + S)/2, \quad Q := (I - S)/2,$$

on the space  $L_N^p(\Gamma, w)$ , where  $\Gamma$  is a smooth curve and  $w$  is an arbitrary Muckenhoupt weight. He also proved that if  $aP + Q$  is semi-Fredholm on  $L_N^p(\Gamma, w)$ , then it is automatically Fredholm on  $L_N^p(\Gamma, w)$ . These results were extended to the case of an arbitrary operator  $A \in \text{alg}(S, PC; L_N^p(\Gamma, w))$  in [12]. The Fredholm theory for singular integral operators with piecewise continuous coefficients on Lebesgue spaces with arbitrary Muckenhoupt weights on arbitrary Carleson curves was accomplished in a series of papers by Böttcher and Yu. Karlovich. It is presented in their monograph [3] (see also the nice survey [4]).

The study of singular integral operators with discontinuous coefficients on generalized Lebesgue spaces with variable exponent was started in [17, 19]. The results of [3] are partially extended to the case of weighted generalized Lebesgue spaces with variable exponent in [13, 14, 15]. Suppose  $\Gamma$  is a Carleson curve satisfying the logarithmic whirl condition (1) at each point  $t \in \Gamma$ ,  $\varrho$  is a Khvedelidze weight, and  $p$  is a variable exponent as in Theorem 1.1. Under these assumptions, a Fredholm criterion for an arbitrary operator  $A$  in the algebra  $\text{alg}(S, PC; L_N^{p(\cdot)}(\Gamma, \varrho))$  is obtained in [14, Theorem 5.1] by using the Allan-Douglas local principle [5, Section 1.35] and the two projections theorem [9]. However, this approach does not allow us to get additional information about semi-Fredholm and Fredholm operators in this algebra. For instance, to obtain an index formula for Fredholm operators in this algebra, we need other means (see, e.g., [15, Section 6]). Following the ideas of [10, 29, 12], in this paper we present a self-contained proof of the following result.

**Theorem 1.2.** *Let  $\Gamma$  be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point  $t \in \Gamma$ , let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). If an operator in the algebra  $\text{alg}(S, PC; L_N^{p(\cdot)}(\Gamma, \varrho))$  is semi-Fredholm, then it is Fredholm.*

The paper is organized as follows. Section 2 contains general results on semi-Fredholm operators. Some auxiliary results on singular integral operators acting on  $L^{p(\cdot)}(\Gamma, \varrho)$  are collected in Section 3. In Section 4, we prove a criterion guaranteeing that  $aP + Q$ , where  $a \in PC(\Gamma)$ , has closed image in  $L^{p(\cdot)}(\Gamma, \varrho)$ . This criterion is intimately related with a Fredholm criterion for  $aP + Q$  proved in [14]. Notice that we are able to prove both results for Carleson Jordan curves which satisfy the additional condition (1). Section 5 contains the proof of the fact that if the operator  $aP + bQ$  is semi-Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , then the coefficients  $a$  and  $b$  are invertible in the algebra  $L_{N \times N}^\infty(\Gamma)$ . In Section 6, we prove that the semi-Fredholmness and Fredholmness of  $aP + bQ$  on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , where  $a$  and  $b$  are piecewise continuous

matrix functions, are equivalent. In Section 7, we extend this result to the sums of products of operators of the form  $aP + bQ$  by using the procedure of linear dilation. Since these sums are dense in  $\text{alg}(S, PC; L_N^{p(\cdot)}(\Gamma, \varrho))$ , Theorem 1.2 follows from stability properties of semi-Fredholm operators.

## 2. GENERAL RESULTS ON SEMI-FREDHOLM AND FREDHOLM OPERATORS

**2.1. The Atkinson and Yood theorems.** For a Banach space  $X$ , let  $\Phi(X)$  be the set of all Fredholm operators on  $X$  and let  $\Phi_+(X)$  (resp.  $\Phi_-(X)$ ) denote the set of all  $n$ -normal (resp.  $d$ -normal) operators  $A \in \mathcal{B}(X)$  such that  $d(A; X) = +\infty$  (resp.  $n(A; X) = +\infty$ ).

**Theorem 2.1.** *Let  $X$  be a Banach space and  $K$  be a compact operator on  $X$ .*

- (a) *If  $A, B \in \Phi(X)$ , then  $AB \in \Phi(X)$  and  $A + K \in \Phi(X)$ .*
- (b) *If  $A, B \in \Phi_{\pm}(X)$ , then  $AB \in \Phi_{\pm}(X)$  and  $A + K \in \Phi_{\pm}(X)$ .*
- (c) *If  $A \in \Phi(X)$  and  $B \in \Phi_{\pm}(X)$ , then  $AB \in \Phi_{\pm}(X)$  and  $BA \in \Phi_{\pm}(X)$ .*

Part (a) is due to Atkinson, parts (b) and (c) were obtained by Yood. For a proof, see e.g. [11, Chap. 4, Sections 6 and 15].

**Theorem 2.2** (see e.g. [11], Chap. 4, Theorem 7.1). *Let  $X$  be a Banach space. An operator  $A \in \mathcal{B}(X)$  is Fredholm if and only if there exists an operator  $R \in \mathcal{B}(X)$  such that  $AR - I$  and  $RA - I$  are compact.*

### 2.2. Stability of semi-Fredholm operators.

**Theorem 2.3** (see e.g. [11], Chap. 4, Theorems 6.4, 15.4). *Let  $X$  be a Banach space.*

- (a) *If  $A \in \Phi(X)$ , then there exists an  $\varepsilon = \varepsilon(A) > 0$  such that  $A + D \in \Phi(X)$  whenever  $\|D\|_{\mathcal{B}(X)} < \varepsilon$ .*
- (b) *If  $A \in \Phi_{\pm}(X)$ , then there exists an  $\varepsilon = \varepsilon(A) > 0$  such that  $A + D \in \Phi_{\pm}(X)$  whenever  $\|D\|_{\mathcal{B}(X)} < \varepsilon$ .*

**Lemma 2.4.** *Let  $X$  be a Banach space. Suppose  $A$  is a semi-Fredholm operator on  $X$  and  $\|A_n - A\|_{\mathcal{B}(X)} \rightarrow 0$  as  $n \rightarrow \infty$ . If the operators  $A_n$  are Fredholm on  $X$  for all sufficiently large  $n$ , then  $A$  is Fredholm, too.*

*Proof.* Assume  $A$  is semi-Fredholm, but not Fredholm. Then either  $A \in \Phi_-(X)$  or  $A \in \Phi_+(X)$ . By Theorem 2.3(b), either  $A_n \in \Phi_-(X)$  or  $A_n \in \Phi_+(X)$  for all sufficiently large  $n$ . That is,  $A_n$  are not Fredholm. This contradicts the hypothesis.  $\square$

We refer to the monograph by Gohberg and Krupnik [11] for a detailed presentation of the theory of semi-Fredholm operators on Banach spaces.

**2.3. Semi-Fredholmness of block operators.** Let a Banach space  $X$  be represented as the direct sum of its subspaces  $X = X_1 \dot{+} X_2$ . Then every operator  $A \in \mathcal{B}(X)$  can be written in the form of an operator matrix

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where  $A_{ij} \in \mathcal{B}(X_j, X_i)$  and  $i, j = 1, 2$ . The following result is stated without proof in [27]. Its proof is given in [28] (see also [23, Theorem 1.12]).

**Theorem 2.5.**

- (a) Suppose  $A_{21}$  is compact. If  $A$  is  $n$ -normal ( $d$ -normal), then  $A_{11}$  (resp.  $A_{22}$ ) is  $n$ -normal (resp.  $d$ -normal).
- (b) Suppose  $A_{12}$  or  $A_{21}$  is compact. If  $A_{11}$  (resp.  $A_{22}$ ) is Fredholm, then  $A_{22}$  (resp.  $A_{11}$ ) is  $n$ -normal,  $d$ -normal, Fredholm if and only if  $A$  has the corresponding property.

### 3. SINGULAR INTEGRALS ON WEIGHTED VARIABLE LEBESGUE SPACES

**3.1. Duality of weighted variable Lebesgue spaces.** Suppose  $\Gamma$  is a rectifiable Jordan curve and  $p : \Gamma \rightarrow (1, \infty)$  is a continuous function. Since  $\Gamma$  is compact, we have

$$1 < \underline{p} := \min_{t \in \Gamma} p(t), \quad \bar{p} := \max_{t \in \Gamma} p(t) < \infty.$$

Define the conjugate exponent  $p^*$  for the exponent  $p$  by

$$p^*(t) := \frac{p(t)}{p(t) - 1} \quad (t \in \Gamma).$$

Suppose  $\varrho$  is a Khvedelidze weight. If  $\varrho \equiv 1$ , then we will write  $L^{p(\cdot)}(\Gamma)$  and  $\|\cdot\|_{p(\cdot)}$  instead of  $L^{p(\cdot)}(\Gamma, 1)$  and  $\|\cdot\|_{p(\cdot),1}$ , respectively.

**Theorem 3.1** (see [21], Theorem 2.1). *If  $f \in L^{p(\cdot)}(\Gamma)$  and  $g \in L^{p^*(\cdot)}(\Gamma)$ , then  $fg \in L^1(\Gamma)$  and*

$$\|fg\|_1 \leq (1 + 1/\underline{p} - 1/\bar{p}) \|f\|_{p(\cdot)} \|g\|_{p^*(\cdot)}.$$

The above Hölder type inequality in the more general setting of Musielak-Orlicz spaces is contained in [24, Theorem 3.13].

**Theorem 3.2.** *The general form of a linear functional on  $L^{p(\cdot)}(\Gamma, \varrho)$  is given by*

$$G(f) = \int_{\Gamma} f(\tau) \overline{g(\tau)} |d\tau| \quad (f \in L^{p(\cdot)}(\Gamma, \varrho)),$$

where  $g \in L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . The norms in the dual space  $[L^{p(\cdot)}(\Gamma, \varrho)]^*$  and in the space  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  are equivalent.

The above result can be extracted from [24, Corollary 13.14]. For the case  $\varrho = 1$ , see also [21, Corollary 2.7].

**3.2. Smirnov classes and Hardy type subspaces.** Let  $\Gamma$  be a rectifiable Jordan curve in the complex plane  $\mathbb{C}$ . We denote by  $D_+$  and  $D_-$  the bounded and unbounded components of  $\mathbb{C} \setminus \Gamma$ , respectively. We orient  $\Gamma$  counter-clockwise. Without loss of generality we assume that  $0 \in D_+$ . A function  $f$  analytic in  $D_+$  is said to be in the Smirnov class  $E^q(D_+)$  ( $0 < q < \infty$ ) if there exists a sequence of rectifiable Jordan curves  $\Gamma_n$  in  $D_+$  tending to the boundary  $\Gamma$  in the sense that  $\Gamma_n$  eventually surrounds each compact subset of  $D_+$  such that

$$(4) \quad \sup_{n \geq 1} \int_{\Gamma_n} |f(z)|^q |dz| < \infty.$$

The Smirnov class  $E^q(D_-)$  is the set of all analytic functions in  $D_- \cup \{\infty\}$  for which (4) holds with some sequence of curves  $\Gamma_n$  tending to the boundary in the sense that every compact subset of  $D_- \cup \{\infty\}$  eventually lies outside  $\Gamma_n$ . We denote by  $E_0^q(D_-)$  the set of functions in  $E^q(D_-)$  which vanish at infinity. The functions in  $E^q(D_\pm)$  have nontangential boundary values almost everywhere on  $\Gamma$  (see, e.g.

[8, Theorem 10.3]). We will identify functions in  $E^q(D_\pm)$  with their nontangential boundary values. The next result is a consequence of the Hölder inequality.

**Lemma 3.3.** *Let  $\Gamma$  be a rectifiable Jordan curve. Suppose  $0 < q_1, q_2, \dots, q_r < \infty$  and  $f_j \in E^{q_j}(D_\pm)$  for all  $j \in \{1, 2, \dots, r\}$ . Then  $f_1 f_2 \dots f_r \in E^q(D_\pm)$ , where  $1/q = 1/q_1 + 1/q_2 + \dots + 1/q_r$ .*

Let  $\mathcal{R}$  denote the set of all rational functions without poles on  $\Gamma$ .

**Theorem 3.4.** *Let  $\Gamma$  be a rectifiable Jordan curve and  $0 < q < \infty$ . If  $f$  belongs to  $E^q(D_\pm) + \mathcal{R}$  and its nontangential boundary values vanish on a subset  $\gamma \subset \Gamma$  of positive measure, then  $f$  vanishes identically in  $D_\pm$ .*

This result follows from the Lusin-Privalov theorem for meromorphic functions (see, e.g. [26, p. 292]).

We refer to the monographs by Duren [8] and Privalov [26] for a detailed exposition of the theory of Smirnov classes over domains with rectifiable boundary.

**Lemma 3.5.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). Then  $P^2 = P$  and  $Q^2 = Q$  on  $L^{p(\cdot)}(\Gamma, \varrho)$ .*

This result follows from Theorem 1.1 and [13, Lemma 6.4].

In view of Lemma 3.5, the Hardy type subspaces  $PL^{p(\cdot)}(\Gamma, \varrho)$ ,  $QL^{p(\cdot)}(\Gamma, \varrho)$ , and  $QL^{p(\cdot)}(\Gamma, \varrho) + \mathbb{C}$  of  $L^{p(\cdot)}(\Gamma, \varrho)$  are well defined. Combining Theorem 1.1 and [13, Lemma 6.9] we obtain the following.

**Lemma 3.6.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). Then*

$$\begin{aligned} E^1(D_+) \cap L^{p(\cdot)}(\Gamma, \varrho) &= PL^{p(\cdot)}(\Gamma, \varrho), \\ E_0^1(D_-) \cap L^{p(\cdot)}(\Gamma, \varrho) &= QL^{p(\cdot)}(\Gamma, \varrho), \\ E^1(D_-) \cap L^{p(\cdot)}(\Gamma, \varrho) &= QL^{p(\cdot)}(\Gamma, \varrho) + \mathbb{C}. \end{aligned}$$

**3.3. Singular integral operators on the dual space.** For a rectifiable Jordan curve  $\Gamma$  we have  $d\tau = e^{i\Theta_\Gamma(\tau)}|d\tau|$  where  $\Theta_\Gamma(\tau)$  is the angle between the positively oriented real axis and the naturally oriented tangent of  $\Gamma$  at  $\tau$  (which exists almost everywhere). Let the operator  $H_\Gamma$  be defined by  $(H_\Gamma\varphi)(t) = e^{-i\Theta_\Gamma(t)}\overline{\varphi(t)}$  for  $t \in \Gamma$ . Note that  $H_\Gamma$  is additive but  $H_\Gamma(\alpha\varphi) = \overline{\alpha}H_\Gamma\varphi$  for  $\alpha \in \mathbb{C}$ . Evidently,  $H_\Gamma^2 = I$ .

From Theorem 1.1 and [13, Lemma 6.6] we get the following.

**Lemma 3.7.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). The adjoint operator of  $S \in \mathcal{B}(L^{p(\cdot)}(\Gamma, \varrho))$  is the operator  $-H_\Gamma S H_\Gamma \in \mathcal{B}(L^{p^*(\cdot)}(\Gamma, \varrho^{-1}))$ .*

**Lemma 3.8.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). Suppose  $a \in L^\infty(\Gamma)$  and  $a^{-1} \in L^\infty(\Gamma)$ .*

(a) *The operator  $aP + Q$  is  $n$ -normal on  $L^{p(\cdot)}(\Gamma, \varrho)$  if and only if the operator  $a^{-1}P + Q$  is  $d$ -normal on  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . In this case*

$$(5) \quad n(aP + Q; L^{p(\cdot)}(\Gamma, \varrho)) = d(a^{-1}P + Q; L^{p^*(\cdot)}(\Gamma, \varrho^{-1})).$$

(b) *The operator  $aP + Q$  is  $d$ -normal on  $L^{p(\cdot)}(\Gamma, \varrho)$  if and only if the operator  $a^{-1}P + Q$  is  $n$ -normal on  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . In this case*

$$d(aP + Q; L^{p(\cdot)}(\Gamma, \varrho)) = n(a^{-1}P + Q; L^{p^*(\cdot)}(\Gamma, \varrho^{-1})).$$

*Proof.* By Theorem 3.2, the space  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  may be identified with the dual space  $[L^{p(\cdot)}(\Gamma, \varrho)]^*$ . Let us prove part (a). The operator  $aP + Q$  is  $n$ -normal on  $L^{p(\cdot)}(\Gamma, \varrho)$  if and only if its adjoint  $(aP + Q)^*$  is  $d$ -normal on the dual space  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  and

$$(6) \quad n(aP + Q; L^{p(\cdot)}(\Gamma, \varrho)) = d((aP + Q)^*; L^{p^*(\cdot)}(\Gamma, \varrho^{-1})).$$

From Theorem 3.2 it follows that

$$(7) \quad (aI)^* = H_\Gamma a H_\Gamma.$$

Combining Lemma 3.7 and (7), we get

$$(8) \quad (aP + Q)^* = H_\Gamma (P + Q a I) H_\Gamma.$$

On the other hand, taking into account Lemma 3.5, it is easy to check that

$$(9) \quad P + Q a I = (I + P a^{-1} Q)(a^{-1} P + Q)(I - Q a^{-1} P)a I,$$

where  $I + P a^{-1} Q$ ,  $I - Q a^{-1} P$ , and  $a I$  are invertible operators on  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . From (8) and (9) it follows that  $(aP + Q)^*$  and  $a^{-1}P + Q$  are  $d$ -normal on the space  $L^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  only simultaneously and

$$(10) \quad d((aP + Q)^*; L^{p^*(\cdot)}(\Gamma, \varrho^{-1})) = d(a^{-1}P + Q; L^{p^*(\cdot)}(\Gamma, \varrho^{-1})).$$

Combining (6) and (10), we arrive at (5). Part (a) is proved. The proof of part (b) is analogous.  $\square$

Denote by  $L_{N \times N}^\infty(\Gamma)$  the algebra of all  $N \times N$  matrix functions with entries in the space  $L^\infty(\Gamma)$ .

**Lemma 3.9.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). Suppose  $a \in L_{N \times N}^\infty(\Gamma)$  and  $a^T$  is the transposed matrix of  $a$ . Then the operator  $P + aQ$  is  $n$ -normal (resp.  $d$ -normal) on  $L_N^{p(\cdot)}(\Gamma, \varrho)$  if and only if the operator  $a^T P + Q$  is  $d$ -normal (resp.  $n$ -normal) on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ .*

*Proof.* In view of Theorem 3.2, the space  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  may be identified with the dual space  $[L_N^{p(\cdot)}(\Gamma, \varrho)]^*$ , and the general form of a linear functional on  $L_N^{p(\cdot)}(\Gamma, \varrho)$  is given by

$$G(f) = \sum_{j=1}^N \int_\Gamma f_j(\tau) \overline{g_j(\tau)} |d\tau|,$$

where  $f = (f_1, \dots, f_N) \in L_N^{p(\cdot)}(\Gamma, \varrho)$  and  $g = (g_1, \dots, g_N) \in L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ , and the norms in  $[L_N^{p(\cdot)}(\Gamma, \varrho)]^*$  and in  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  are equivalent. It is easy to see that  $(aI)^* = H_\Gamma a^T H_\Gamma$ , where  $H_\Gamma$  is defined on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  elementwise. From Lemma 3.7 it follows that  $P^* = H_\Gamma Q H_\Gamma$  and  $Q^* = H_\Gamma P H_\Gamma$  on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . Then

$$(11) \quad (P + aQ)^* = H_\Gamma (P a^T I + Q) H_\Gamma.$$

On the other hand, it is easy to see that

$$(12) \quad Pa^T I + Q = (I + Pa^T Q)(a^T P + Q)(I - Qa^T P),$$

where the operators  $I + Pa^T Q$  and  $I - Qa^T P$  are invertible on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . From (11) and (12) it follows that  $(P + aQ)^*$  and  $a^T P + Q$  are  $n$ -normal (resp.  $d$ -normal) on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$  only simultaneously. This implies the desired statement.  $\square$

#### 4. CLOSEDNESS OF THE IMAGE OF $aP + Q$ IN THE SCALAR CASE

##### 4.1. Functions in $L^{p(\cdot)}(\Gamma, \varrho)$ are better than integrable if $S$ is bounded.

**Lemma 4.1.** *Suppose  $\Gamma$  is a Carleson Jordan curve and  $p : \Gamma \rightarrow (1, \infty)$  is a continuous function satisfying (2). If  $\varrho$  is a Khvedelidze weight satisfying (3), then there exists an  $\varepsilon > 0$  such that  $L^{p(\cdot)}(\Gamma, \varrho)$  is continuously embedded in  $L^{1+\varepsilon}(\Gamma)$ .*

*Proof.* If (3) holds, then there exists a number  $\varepsilon > 0$  such that

$$0 < (1/p(t_k) + \lambda_k)(1 + \varepsilon) < 1 \quad \text{for all } k \in \{1, \dots, m\}.$$

Hence, by Theorem 1.1, the operator  $S$  is bounded on  $L^{p(\cdot)/(1+\varepsilon)}(\Gamma, \varrho^{1+\varepsilon})$ . In that case the operator  $\varrho^{1+\varepsilon} S \varrho^{-1-\varepsilon} I$  is bounded on  $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$ . Obviously, the operator  $V$  defined by  $(Vg)(t) = tg(t)$  is bounded on  $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$ , and

$$((AV - VA)g)(t) = \frac{\varrho^{1+\varepsilon}(t)}{\pi i} \int_{\Gamma} \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)} d\tau.$$

Since  $AV - VA$  is bounded on  $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$ , there exists a constant  $C > 0$  such that

$$\left| \int_{\Gamma} \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)} d\tau \right| \|\varrho^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} = \left\| \varrho^{1+\varepsilon} \int_{\Gamma} \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)} d\tau \right\|_{p(\cdot)/(1+\varepsilon)} \leq C \|g\|_{p(\cdot)/(1+\varepsilon)}$$

for all  $g \in L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$ . Since  $\varrho(\tau) > 0$  a.e. on  $\Gamma$ , we have  $\|\varrho^{1+\varepsilon}\|_{p(\cdot)/(1+\varepsilon)} > 0$ . Hence

$$\Lambda(g) = \int_{\Gamma} \frac{g(\tau)}{\varrho^{1+\varepsilon}(\tau)} e^{i\Theta_{\Gamma}(\tau)} |d\tau|$$

is a bounded linear functional on  $L^{p(\cdot)/(1+\varepsilon)}(\Gamma)$ . From Theorem 3.2 it follows that  $\varrho^{-1-\varepsilon} \in L^{[p(\cdot)/(1+\varepsilon)]^*}(\Gamma)$ , where

$$\left( \frac{p(t)}{1+\varepsilon} \right)^* = \frac{p(t)}{p(t) - (1+\varepsilon)}$$

is the conjugate exponent for  $p(\cdot)/(1+\varepsilon)$ . By Theorem 3.1,

$$(13) \quad \int_{\Gamma} |f(\tau)|^{1+\varepsilon} |d\tau| \leq C_{p(\cdot), \varepsilon} \| |f|^{1+\varepsilon} \varrho^{1+\varepsilon} \|_{p(\cdot)/(1+\varepsilon)} \|\varrho^{-1-\varepsilon}\|_{[p(\cdot)/(1+\varepsilon)]^*}.$$

It is easy to see that

$$(14) \quad \| |f|^{1+\varepsilon} \varrho^{1+\varepsilon} \|_{p(\cdot)/(1+\varepsilon)} = \| f \varrho \|_{p(\cdot)}^{1+\varepsilon} = \| f \|_{p(\cdot), \varrho}^{1+\varepsilon}.$$

From (13) and (14) it follows that  $\|f\|_{1+\varepsilon} \leq C_{p(\cdot), \varepsilon, \varrho} \|f\|_{p(\cdot), \varrho}$  for all  $f \in L^{p(\cdot)}(\Gamma, \varrho)$ , where  $C_{p(\cdot), \varepsilon, \varrho} := (C_{p(\cdot), \varepsilon} \|\varrho^{-1-\varepsilon}\|_{[p(\cdot)/(1+\varepsilon)]^*})^{1/(1+\varepsilon)} < \infty$ .  $\square$

#### 4.2. Criterion for Fredholmness of $aP + Q$ in the scalar case.

**Theorem 4.2** (see [14], Theorem 3.3). *Let  $\Gamma$  be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point  $t \in \Gamma$ , let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). Suppose  $a \in PC(\Gamma)$ . The operator  $aP + Q$  is Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$  if and only if  $a(t \pm 0) \neq 0$  and*

$$(15) \quad -\frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| + \frac{1}{p(t)} + \lambda(t) \notin \mathbb{Z}$$

for all  $t \in \Gamma$ , where

$$\lambda(t) := \begin{cases} \lambda_k, & \text{if } t = t_k, \quad k \in \{1, \dots, m\}, \\ 0, & \text{if } t \notin \Gamma \setminus \{t_1, \dots, t_m\}. \end{cases}$$

The necessity portion of this result was obtained in [13, Theorem 8.1] for spaces with variable exponents satisfying (2) under the assumption that  $S$  is bounded on  $L^{p(\cdot)}(\Gamma, w)$ , where  $\Gamma$  is an arbitrary rectifiable Jordan curve and  $w$  is an arbitrary weight (not necessarily power). The sufficiency portion follows from [13, Lemma 7.1] and Theorem 1.1 (see [14] for details). The restriction (1) comes up in the proof of the sufficiency portion because under this condition one can guarantee the boundedness of the weighted operator  $wSw^{-1}I$ , where  $w(\tau) = |(t-\tau)^\gamma|$  and  $\gamma \in \mathbb{C}$ . If  $\Gamma$  does not satisfy (1), then the weight  $w$  is not equivalent to a Khvedelidze weight and Theorem 1.1 is not applicable to the operator  $wSw^{-1}I$ , that is, a more general result than Theorem 1.1 is needed to treat the case of arbitrary Carleson curves. As far as we know, such a result is not known in the case of variable exponents. For a constant exponent  $p$ , the result of Theorem 4.2 (for arbitrary Muckenhoupt weights) is proved in [2] (see also [3, Proposition 7.3] for the case of arbitrary Muckenhoupt weights and arbitrary Carleson curves).

#### 4.3. Criterion for the closedness of the image of $aP + Q$ .

**Theorem 4.3.** *Let  $\Gamma$  be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point  $t \in \Gamma$ , let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). Suppose  $a \in PC(\Gamma)$  has finitely many jumps and  $a(t \pm 0) \neq 0$  for all  $t \in \Gamma$ . Then the image of  $aP + Q$  is closed in  $L^{p(\cdot)}(\Gamma, \varrho)$  if and only if (15) holds for all  $t \in \Gamma$ .*

*Proof.* The idea of the proof is borrowed from [3, Proposition 7.16]. The sufficiency part follows from Theorem 4.2. Let us prove the necessity part. Assume that  $a(t \pm 0) \neq 0$  for all  $t \in \Gamma$ . Since the number of jumps, that is, the points  $t \in \Gamma$  at which  $a(t-0) \neq a(t+0)$ , is finite, it is clear that

$$\begin{aligned} & -\frac{1}{2\pi} \arg \frac{a(t-0)}{a(t+0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t-0)}{a(t+0)} \right| + \frac{1}{1+\varepsilon} \notin \mathbb{Z}, \\ & -\frac{1}{2\pi} \arg \frac{a(t+0)}{a(t-0)} + \frac{\delta(t)}{2\pi} \log \left| \frac{a(t+0)}{a(t-0)} \right| + \frac{1}{1+\varepsilon} \notin \mathbb{Z} \end{aligned}$$

for all  $t \in \Gamma$  and all sufficiently small  $\varepsilon > 0$ . By Theorem 4.2, the operators  $aP + Q$  and  $a^{-1}P + Q$  are Fredholm on the Lebesgue space  $L^{1+\varepsilon}(\Gamma)$  whenever  $\varepsilon > 0$  is sufficiently small. From Lemma 4.1 it follows that we can pick  $\varepsilon_0 > 0$  such that

$$L^{p(\cdot)}(\Gamma, \varrho) \subset L^{1+\varepsilon_0}(\Gamma), \quad L^{p^*(\cdot)}(\Gamma, \varrho^{-1}) \subset L^{1+\varepsilon_0}(\Gamma)$$

and  $aP + Q$ ,  $a^{-1}P + Q$  are Fredholm on  $L^{1+\varepsilon_0}(\Gamma)$ . Then

$$(16) \quad n(aP + Q; L^{p(\cdot)}(\Gamma, \varrho)) \leq n(aP + Q; L^{1+\varepsilon_0}(\Gamma)) < \infty,$$

and taking into account Lemma 3.8(b),

$$(17) \quad \begin{aligned} d(aP + Q; L^{p(\cdot)}(\Gamma, \varrho)) &= n(a^{-1}P + Q; L^{p^*(\cdot)}(\Gamma, \varrho^{-1})) \\ &\leq n(a^{-1}P + Q; L^{1+\varepsilon_0}(\Gamma)) < \infty. \end{aligned}$$

If (15) does not hold, then  $aP + Q$  is not Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$  in view of Theorem 4.2. From this fact and (16)–(17) we conclude that the image of  $aP + Q$  is not closed in  $L^{p(\cdot)}(\Gamma, \varrho)$ , which contradicts the hypothesis.  $\square$

## 5. NECESSARY CONDITION FOR SEMI-FREDHOLMNESS OF $aP + bQ$ . THE MATRIX CASE

**5.1. Two lemmas on approximation of measurable matrix functions.** Let the algebra  $L_{N \times N}^\infty(\Gamma)$  be equipped with the norm

$$\|a\|_{L_{N \times N}^\infty(\Gamma)} := N \max_{1 \leq i, j \leq N} \|a_{ij}\|_{L^\infty(\Gamma)}.$$

**Lemma 5.1** (see [23], Lemma 3.4). *Let  $\Gamma$  be a rectifiable Jordan curve. Suppose  $a$  is a measurable  $N \times N$  matrix function on  $\Gamma$  such that  $a^{-1} \notin L_{N \times N}^\infty(\Gamma)$ . Then for every  $\varepsilon > 0$  there exists a matrix function  $a_\varepsilon \in L_{N \times N}^\infty(\Gamma)$  such that  $\|a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} < \varepsilon$  and the matrix function  $a - a_\varepsilon$  degenerates on a subset  $\gamma \subset \Gamma$  of positive measure.*

**Lemma 5.2** (see [23], Lemma 3.6). *Let  $\Gamma$  be a rectifiable Jordan curve. If  $a$  belongs to  $L_{N \times N}^\infty(\Gamma)$ , then for every  $\varepsilon > 0$  there exists an  $a_\varepsilon \in L_{N \times N}^\infty(\Gamma)$  such that  $\|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} < \varepsilon$  and  $a_\varepsilon^{-1} \in L_{N \times N}^\infty(\Gamma)$ .*

### 5.2. Necessary condition for $d$ -normality of $aP + Q$ and $P + aQ$ .

**Lemma 5.3.** *Suppose  $\Gamma$  is a Carleson Jordan curve,  $p : \Gamma \rightarrow (1, \infty)$  is a continuous function satisfying (2), and  $\varrho$  is a Khvedelidze weight satisfying (3). If  $a \in L_{N \times N}^\infty(\Gamma)$  and at least one of the operators  $aP + Q$  or  $P + aQ$  is  $d$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , then  $a^{-1} \in L_{N \times N}^\infty(\Gamma)$ .*

*Proof.* This lemma is proved by analogy with [23, Theorem 3.13]. For definiteness, let us consider the operator  $P + aQ$ . Assume that  $a^{-1} \notin L_{N \times N}^\infty(\Gamma)$ . By Lemma 5.1, for every  $\varepsilon > 0$  there exists an  $a_\varepsilon \in L_{N \times N}^\infty(\Gamma)$  such that  $\|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} < \varepsilon$  and  $a_\varepsilon$  degenerates on a subset  $\gamma \subset \Gamma$  of positive measure. We have

$$\|(P + aQ) - (P + a_\varepsilon Q)\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} \leq \|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} \|Q\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} = O(\varepsilon)$$

as  $\varepsilon \rightarrow 0$ . Hence there is an  $\varepsilon > 0$  such that  $P + a_\varepsilon Q$  is  $d$ -normal together with  $P + aQ$  due to Theorem 2.3. Since the image of the operator  $P + a_\varepsilon Q$  is a subspace of finite codimension in  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , it has a nontrivial intersection with any infinite-dimensional linear manifold contained in  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . In particular, the image of  $P + a_\varepsilon Q$  has a nontrivial intersection with linear manifolds  $M_j$ ,  $j \in \{1, \dots, N\}$ , of those vector-functions, the  $j$ -th component of which is a polynomial of  $1/z$  vanishing at infinity and all the remaining components are identically zero. That is, there exist

$$\psi_j^+ \in PL_N^{p(\cdot)}(\Gamma, \varrho), \quad \psi_j^- \in QL_N^{p(\cdot)}(\Gamma, \varrho), \quad h_j \in M_j, \quad h_j \not\equiv 0$$

such that  $\psi_j^+ + a_\varepsilon \psi_j^- = h_j$  for all  $j \in \{1, \dots, N\}$ . Consider the  $N \times N$  matrix functions

$$\Psi_+ := [\psi_1^+, \psi_2^+, \dots, \psi_N^+], \quad \Psi_- := [\psi_1^-, \psi_2^-, \dots, \psi_N^-], \quad H := [h_1, h_2, \dots, h_N],$$

where  $\psi_j^+$ ,  $\psi_j^-$ , and  $h_j$  are taken as columns. Then  $H - \Psi_+ = a_\varepsilon \Psi_-$ . Therefore,

$$\det(H - \Psi_+) = \det a_\varepsilon \det \Psi_- \quad \text{a.e. on } \Gamma.$$

The left-hand side of this equality is a meromorphic function having a pole at zero of at least  $N$ -th order. Thus, it is not identically zero in  $D_+$ .

On the other hand, each entry of  $H - \Psi_+$  belongs to

$$PL^{p(\cdot)}(\Gamma, \varrho) + \mathcal{R} \subset E^1(D_+) + \mathcal{R}$$

(see Lemma 3.6). Hence, by Lemma 3.3,  $\det(H - \Psi_+) \in E^{1/N}(D_+) + \mathcal{R}$  and  $\det(H - \Psi_+)$  degenerates on  $\gamma$  because  $a_\varepsilon$  degenerates on  $\gamma$ . In view of Theorem 3.4,  $\det(H - \Psi_+)$  vanishes identically in  $D_+$ . This is a contradiction. Thus,  $a^{-1}$  belongs to  $L_{N \times N}^\infty(\Gamma)$ .  $\square$

### 5.3. Necessary condition for semi-Fredholmness of $aP + bQ$ .

**Theorem 5.4.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). If the coefficients  $a$  and  $b$  belong to  $L_{N \times N}^\infty(\Gamma)$  and the operator  $aP + bQ$  is semi-Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , then  $a^{-1}, b^{-1} \in L_{N \times N}^\infty(\Gamma)$ .*

*Proof.* The proof is analogous to the proof of [23, Theorem 3.18]. Suppose  $aP + bQ$  is  $d$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . By Lemma 5.2, for every  $\varepsilon > 0$  there exist  $a_\varepsilon \in L_{N \times N}^\infty(\Gamma)$  such that  $a_\varepsilon^{-1} \in L_{N \times N}^\infty(\Gamma)$  and  $\|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} < \varepsilon$ . Since

$$\|(aP + bQ) - (a_\varepsilon P + bQ)\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} \leq \|a - a_\varepsilon\|_{L_{N \times N}^\infty(\Gamma)} \|P\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} = O(\varepsilon)$$

as  $\varepsilon \rightarrow 0$ , from Theorem 2.3 it follows that  $\varepsilon > 0$  can be chosen so small that  $a_\varepsilon P + bQ$  is  $d$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , too. Since  $a_\varepsilon^{-1} \in L_{N \times N}^\infty(\Gamma)$ , the operator  $a_\varepsilon I$  is invertible on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . From Theorem 2.1 it follows that the operator  $P + a_\varepsilon^{-1}bQ = a_\varepsilon^{-1}(a_\varepsilon P + bQ)$  is  $d$ -normal. By Lemma 5.3,  $b^{-1}a_\varepsilon$  belongs to  $L_{N \times N}^\infty(\Gamma)$ . Hence  $b^{-1} = b^{-1}a_\varepsilon a_\varepsilon^{-1} \in L_{N \times N}^\infty(\Gamma)$ .

Furthermore,  $b^{-1}aP + Q = b^{-1}(aP + bQ)$  and the operator  $b^{-1}aP + Q$  is  $d$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . By Lemma 5.3,  $a^{-1}b \in L_{N \times N}^\infty(\Gamma)$ . Then  $a^{-1} = a^{-1}bb^{-1}$  belongs to  $L_{N \times N}^\infty(\Gamma)$ . That is, we have shown that if  $aP + bQ$  is  $d$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , then  $a^{-1}, b^{-1} \in L_{N \times N}^\infty(\Gamma)$ .

If  $aP + bQ$  is  $n$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , then arguing as above, we conclude that the operator  $P + a_\varepsilon^{-1}bQ$  is  $n$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . By Lemma 3.9, the operator  $(a_\varepsilon^{-1}b)^T P + Q$  is  $d$ -normal on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . From Lemma 5.3 it follows that  $[(a_\varepsilon^{-1}b)^T]^{-1} \in L_{N \times N}^\infty(\Gamma)$ . Therefore,  $b^{-1} = (a_\varepsilon^{-1})^{-1}a_\varepsilon^{-1} \in L_{N \times N}^\infty(\Gamma)$ . Furthermore,  $b^{-1}aP + Q = b^{-1}(aP + bQ)$  and the operator  $b^{-1}aP + Q = b^{-1}(aP + bQ)$  is  $n$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . From Lemma 3.9 we get that the operator  $P + (b^{-1}a)^T Q$  is  $d$ -normal on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ . Applying Lemma 5.3 to the operator  $P + (b^{-1}a)^T Q$  acting on  $L_N^{p^*(\cdot)}(\Gamma, \varrho^{-1})$ , we obtain  $a^{-1}b \in L_{N \times N}^\infty(\Gamma)$ . Thus  $a^{-1} = a^{-1}bb^{-1} \in L_{N \times N}^\infty(\Gamma)$ .  $\square$

## 6. SEMI-FREDHOLMNESS AND FREDHOLMNESS OF $aP + bQ$ ARE EQUIVALENT

**6.1. Decomposition of piecewise continuous matrix functions.** Denote by  $PC^0(\Gamma)$  the set of all piecewise continuous functions  $a$  which have only a finite number of jumps and satisfy  $a(t-0) = a(t)$  for all  $t \in \Gamma$ . Let  $C_{N \times N}(\Gamma)$  and  $PC_{N \times N}^0(\Gamma)$  denote the sets of  $N \times N$  matrix functions with continuous entries and with entries in  $PC^0(\Gamma)$ , respectively. A matrix function  $a \in PC_{N \times N}(\Gamma)$  is said to be nonsingular if  $\det a(t \pm 0) \neq 0$  for all  $t \in \Gamma$ .

**Lemma 6.1** (see [6], Chap. VII, Lemma 2.2). *Suppose  $\Gamma$  is a rectifiable Jordan curve. If a matrix function  $f \in PC_{N \times N}^0(\Gamma)$  is nonsingular, then there exist an upper-triangular nonsingular matrix function  $g \in PC_{N \times N}^0(\Gamma)$  and nonsingular matrix functions  $c_1, c_2 \in C_{N \times N}(\Gamma)$  such that  $f = c_1 g c_2$ .*

### 6.2. Compactness of commutators.

**Lemma 6.2.** *Let  $\Gamma$  be a Carleson Jordan curve, let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). If  $c$  belongs to  $C_{N \times N}(\Gamma)$ , then the commutators  $cP - P c I$  and  $cQ - Q c I$  are compact on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ .*

This statement follows from Theorem 1.1 and [13, Lemma 6.5].

### 6.3. Equivalence of semi-Fredholmness and Fredholmness of $aP + bQ$ .

**Theorem 6.3.** *Let  $\Gamma$  be a Carleson Jordan curve satisfying the logarithmic whirl condition (1) at each point  $t \in \Gamma$ , let  $p : \Gamma \rightarrow (1, \infty)$  be a continuous function satisfying (2), and let  $\varrho$  be a Khvedelidze weight satisfying (3). If  $a, b \in PC_{N \times N}^0(\Gamma)$ , then  $aP + bQ$  is semi-Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$  if and only if it is Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ .*

*Proof.* The idea of the proof is borrowed from [29, Theorem 3.1]. Only the necessity portion of the theorem is nontrivial. If  $aP + bQ$  is semi-Fredholm, then  $a$  and  $b$  are nonsingular by Theorem 5.4. Hence  $b^{-1}a$  is nonsingular. In view of Lemma 6.1, there exist an upper-triangular nonsingular matrix function  $g \in PC_{N \times N}^0(\Gamma)$  and continuous nonsingular matrix functions  $c_1, c_2$  such that  $b^{-1}a = c_1 g c_2$ . It is easy to see that

$$(18) \quad aP + bQ = bc_1 [(gP + Q)(Pc_2 I + Qc_1^{-1} I) + g(c_2 P - P c_2 I) + (c_1^{-1} Q - Q c_1^{-1} I)].$$

From Lemma 6.2 it follows that the operators  $c_2 P - P c_2 I$  and  $c_1^{-1} Q - Q c_1^{-1} I$  are compact on  $L_N^{p(\cdot)}(\Gamma, \varrho)$  and

$$(Pc_2 I + Qc_1^{-1} I)(c_2^{-1} P + c_1 Q) = I + K_1, \quad (c_2^{-1} P + c_1 Q)(Pc_2 I + Qc_1^{-1} I) = I + K_2,$$

where  $K_1$  and  $K_2$  are compact operators on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . In view of these equalities, by Theorem 2.2, the operator  $Pc_2 I + Qc_1^{-1} I$  is Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . Obviously, the operator  $bc_1 I$  is invertible because  $bc_1$  is nonsingular. From (18) and Theorem 2.1 it follows that  $aP + bQ$  is  $n$ -normal,  $d$ -normal, Fredholm if and only if  $gP + Q$  has the corresponding property.

Let  $g_j$ ,  $j \in \{1, \dots, N\}$ , be the elements of the main diagonal of the upper-triangular matrix function  $g$ . Since  $g$  is nonsingular, all  $g_j$  are nonsingular, too. Assume for definiteness that  $gP + Q$  is  $n$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . By Theorem 2.5(a), the operator  $g_1 P + Q$  is  $n$ -normal on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . Hence the image of  $g_1 P + Q$  is closed. From Theorem 4.3 it follows that (15) is fulfilled with  $g_1$  in place of  $a$ .

Therefore, the operator  $g_1 P + Q$  is Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$  due to Theorem 4.2. Applying Theorem 2.5(b), we deduce that the operator  $g^{(1)}P + Q$  is  $n$ -normal on  $L_{N-1}^{p(\cdot)}(\Gamma, \varrho)$ , where  $g^{(1)}$  is the  $(N-1) \times (N-1)$  upper-triangular nonsingular matrix function obtained from  $g$  by deleting the first column and the first row. Arguing as before with  $g^{(1)}$  in place of  $g$ , we conclude that  $g_2 P + Q$  is Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$  and  $g^{(2)}P + Q$  is  $n$ -normal on  $L_{N-2}^{p(\cdot)}(\Gamma, \varrho)$ , where  $g^{(2)}$  is the  $(N-2) \times (N-2)$  upper-triangular nonsingular matrix function obtained from  $g^{(1)}$  by deleting the first column and the first row. Repeating this procedure  $N$  times, we can show that all operators  $g_j P + Q$ ,  $j \in \{1, \dots, N\}$ , are Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$ .

If the operator  $gP + Q$  is  $d$ -normal, then we can prove in a similar fashion that all operators  $g_j P + Q$ ,  $j \in \{1, \dots, N\}$ , are Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$ . In this case we start with  $g_N$  and delete the last column and the last row of the matrix  $g^{(j-1)}$  on the  $j$ -th step (we assume that  $g^{(0)} = g$ ).

Since all operators  $g_j P + Q$  are Fredholm on  $L^{p(\cdot)}(\Gamma, \varrho)$ , from Theorem 2.5(b) we obtain that the operator  $gP + Q$  is Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . Hence  $aP + bQ$  is Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ , too.  $\square$

## 7. SEMI-FREDHOLMNESS AND FREDHOLMNESS ARE EQUIVALENT FOR ARBITRARY OPERATORS IN $\text{alg}(S, PC, L_N^{p(\cdot)}(\Gamma, \varrho))$

**7.1. Linear dilation.** The following statement shows that the semi-Fredholmness of an operator in a dense subalgebra of  $\text{alg}(S, PC, L_N^{p(\cdot)}(\Gamma, \varrho))$  is equivalent to the semi-Fredholmness of a simpler operator  $aP + bQ$  with coefficients of  $a, b$  of larger size.

**Lemma 7.1.** *Suppose  $\Gamma$  is a Carleson Jordan curve,  $p : \Gamma \rightarrow (1, \infty)$  is a continuous function satisfying (2), and  $\varrho$  is a Khvedelidze weight satisfying (3). Let*

$$A = \sum_{i=1}^k A_{i1} A_{i2} \dots A_{ir},$$

where  $A_{ij} = a_{ij}P + b_{ij}Q$  and all  $a_{ij}, b_{ij}$  belong to  $PC_{N \times N}^0(\Gamma)$ . Then there exist functions  $a, b \in PC_{D \times D}^0(\Gamma)$ , where  $D := N(k(r+1)+1)$ , such that  $A$  is  $n$ -normal ( $d$ -normal, Fredholm) on  $L_N^{p(\cdot)}(\Gamma, \varrho)$  if and only if  $aP + bQ$  is  $n$ -normal (resp.  $d$ -normal, Fredholm) on  $L_D^{p(\cdot)}(\Gamma, \varrho)$ .

*Proof.* The idea of the proof is borrowed from [10] (see also [1, Theorem 12.15]). Denote by  $O_s$  and  $I_s$  the  $s \times s$  zero and identity matrix, respectively. For  $\ell = 1, \dots, r$ , let  $B_\ell$  be the  $kN \times kN$  matrix

$$B_\ell = \text{diag}(A_{1\ell}, A_{2\ell}, \dots, A_{k\ell}),$$

then define the  $kN(r+1) \times kN(r+1)$  matrix  $Z$  by

$$Z = \begin{bmatrix} I_{kN} & B_1 & O_{kN} & \dots & O_{kN} \\ O_{kN} & I_{kN} & B_2 & \dots & O_{kN} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{kN} & O_{kN} & O_{kN} & \dots & B_r \\ O_{kN} & O_{kN} & O_{kN} & \dots & I_{kN} \end{bmatrix}.$$

Put

$$X := \text{column}(\underbrace{O_N, \dots, O_N}_{kr}, \underbrace{-I_N, \dots, -I_N}_k), \quad Y := (\underbrace{I_N, \dots, I_N}_k, \underbrace{O_N, \dots, O_N}_{kr}).$$

Define also  $M_0 = (\underbrace{I_N, \dots, I_N}_k)$  and for  $\ell \in \{1, \dots, r\}$ , let

$$M_\ell := (A_{11}A_{12} \dots A_{1\ell}, A_{21}A_{22} \dots A_{2\ell}, \dots, A_{k1}A_{k2} \dots A_{k\ell}).$$

Finally, put

$$W := (M_0, M_1, \dots, M_r).$$

It can be verified straightforwardly that

$$(19) \quad \begin{bmatrix} I_{kN(r+1)} & O \\ W & I_N \end{bmatrix} \begin{bmatrix} I_{kN(r+1)} & O \\ O & A \end{bmatrix} \begin{bmatrix} Z & X \\ O & I_N \end{bmatrix} = \begin{bmatrix} Z & X \\ Y & O_N \end{bmatrix}.$$

It is clear that the outer terms on the left-hand side of (19) are invertible. Hence the middle factor of (19) and the right-hand side of (19) are  $n$ -normal ( $d$ -normal, Fredholm) only simultaneously in view of Theorem 2.1. By Theorem 2.5(b), the operator  $A$  is  $n$ -normal ( $d$ -normal, Fredholm) if and only if the middle factor of (19) has the corresponding property. Finally, note that the left-hand side of (19) has the form  $aP + bQ$ , where  $a, b \in PC_{D \times D}^0(\Gamma)$ .  $\square$

**7.2. Proof of Theorem 1.2.** Obviously, for every  $f \in PC(\Gamma)$  there exists a sequence  $f_n \in PC^0(\Gamma)$  such that  $\|f - f_n\|_{L^\infty(\Gamma)} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, for each operator  $\alpha P + \beta Q$ , where  $\alpha = (\alpha_{rs})_{r,s=1}^N$ ,  $\beta = (\beta_{rs})_{r,s=1}^N$  and  $\alpha_{rs}, \beta_{rs} \in PC(\Gamma)$  for all  $r, s \in \{1, \dots, N\}$ , there exist sequences  $\alpha^{(n)} = (\alpha_{rs}^{(n)})_{r,s=1}^N$ ,  $\beta^{(n)} = (\beta_{rs}^{(n)})_{r,s=1}^N$  with  $\alpha_{rs}^{(n)}, \beta_{rs}^{(n)} \in PC^0(\Gamma)$  for all  $r, s \in \{1, \dots, N\}$  such that

$$\begin{aligned} & \|(\alpha P + \beta Q) - (\alpha^{(n)} P + \beta^{(n)} Q)\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} \\ & \leq N \max_{1 \leq r, s \leq N} \|\alpha_{rs} - \alpha_{rs}^{(n)}\|_{L^\infty(\Gamma)} \|P\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} \\ & \quad + N \max_{1 \leq r, s \leq N} \|\beta_{rs} - \beta_{rs}^{(n)}\|_{L^\infty(\Gamma)} \|Q\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} = o(1) \end{aligned}$$

as  $n \rightarrow \infty$ .

Let  $A \in \text{alg}(S, PC; L_N^{p(\cdot)}(\Gamma, \varrho))$ . Then there exists a sequence of operators  $A^{(n)}$  of the form  $\sum_{i=1}^k A_{i1}^{(n)} A_{i2}^{(n)} \dots A_{ir}^{(n)}$ , where  $A_{ij}^{(n)} = a_{ij}^{(n)} P + b_{ij}^{(n)} Q$  and  $a_{ij}^{(n)}, b_{ij}^{(n)}$  belong to  $PC_{N \times N}(\Gamma)$ , such that  $\|A - A^{(n)}\|_{\mathcal{B}(L_N^{p(\cdot)}(\Gamma, \varrho))} \rightarrow 0$  as  $n \rightarrow \infty$ . In view of what has been said above, without loss of generality, we can assume that all matrix functions  $a_{ij}^{(n)}, b_{ij}^{(n)}$  belong to  $PC_{N \times N}^0(\Gamma)$ .

If  $A$  is semi-Fredholm, then for all sufficiently large  $n$ , the operators  $A^{(n)}$  are semi-Fredholm by Theorem 2.3. From Lemma 7.1 it follows that for every semi-Fredholm operator  $\sum_{i=1}^k A_{i1}^{(n)} A_{i2}^{(n)} \dots A_{ir}^{(n)}$  there exist  $a^{(n)}, b^{(n)} \in PC_{D \times D}^0(\Gamma)$ , where  $D := N(k(r+1)+1)$ , such that  $a^{(n)}P + b^{(n)}Q$  is semi-Fredholm on  $L_D^{p(\cdot)}(\Gamma, \varrho)$ . By Theorem 6.3,  $a^{(n)}P + b^{(n)}Q$  is Fredholm on  $L_D^{p(\cdot)}(\Gamma, \varrho)$ . Applying Lemma 7.1 again, we conclude that  $\sum_{i=1}^k A_{i1}^{(n)} A_{i2}^{(n)} \dots A_{ir}^{(n)}$  is Fredholm on  $L_N^{p(\cdot)}(\Gamma, \varrho)$ . Thus, for all sufficiently large  $n$ , the operators  $A^{(n)}$  are Fredholm. Lemma 2.4 yields that  $A$  is Fredholm.  $\square$

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